ORDRE DES INGÉNIEURS DU QUÉBEC

MAY 2014 SESSION

Open-book examination
Calculators : only authorized models
Duration : 3 hours

14-PH-B2 ELECTRO-OPTICAL ENGINEERING

Question 1 (10 points). Lasing threshold.

Consider laser cavity of length $L=10cm$ with a ruby crystal as an active media. One of the cavity mirrors is highly reflective $R_1=0.999$, while another one is semitransparent allowing *5%* of the incoming light to be transmitted out of the laser cavity. Absorption coefficient of a ruby crystal in the ground state is $\alpha=0.5cm^{-1}$ (at the wavelength of laser emission). Find what portion of the chromium atoms in the ruby crystal has to be converted into the excited state in order for the laser to start its operation. To solve this problem, neglect scattering losses in the laser cavity.

Question 2 (20 points). Circular refraction in the atmosphere of Venus.

On Venus, the atmosphere consists mostly of CO_2 gas (96.5%) with molecular polarizability of $\alpha_{CO_2}=2.507\cdot 10^{-24}cm^3$ (cgs units), and molecular mass $m_{CO_2}=7.31\cdot 10^{-23}g$. The pressure at the surface of Venus is $p_0=90\ atm\ (9000\ kPa)$, and the temperature is $T_0=480^oC\ (750K)$. The standard acceleration due to free fall on Venus is $g_V=8.87\,m/s^2$, while the planetary radius is $r_V=6052$ km. Due to significant pressure at the Venus surface, the gradient of the refractive index of the planet atmosphere can be considerable. This leads to an interesting optical phenomenon of trapping the light emitted parallel to the planet surface. Particularly, we are interested in finding the radius of curvature of light r_{light} sent parallel to the Venus surface, and demonstrating that it is smaller than the planet radius $|r_{light}|<r_V$. In order to demonstrate that, proceed through the following steps:

i) (5 points). Find concentration $N_{CO_2}(h)$ of the CO_2 gas molecules as a function of the elevation h above the planet surface by using a well known result from statistical mechanics stating that:

$$N_{CO_2}(h)\approx N_0\exp\left(-\frac{U(h)}{k_bT_0}\right), (1)$$

where $U(h)$ is the gravitational potential energy of a single CO_2 molecule at elevation h above the planet surface, $k_b=1.3806488\cdot 10^{-23}\mathrm{m^2kg\,s^{-2}K^{-1}}$ is Boltzmann constant, and $N_0=p_0/(k_bT_0)$ assuming that CO_2 gas behaves as an ideal gas at the planet surface. By substituting the values of all the physical parameters into (1), verify that a much simpler linear approximation of (1) (Taylor expansion to the first order in h) can be used when $h \ll 16km$.

ii) (5 points). Find dependence of the atmospheric refractive index $n(h)$ on the elevation above the planet surface. To do that, use the results of (i) and Clausis-Mossotti formula that relates dielectric constant of a polarizable dielectric media to the concentration and polarizability of the individual molecules:

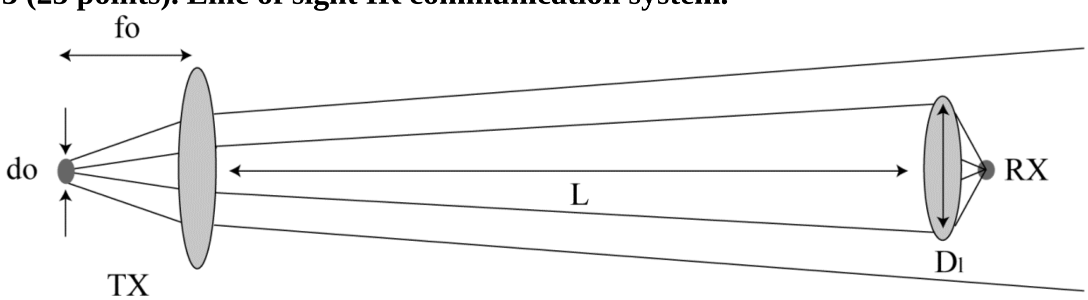
$$\frac{\varepsilon - 1}{\varepsilon + 2} \underset{\substack{CGS \\ units}}{=} \frac{4\pi}{3} \alpha_{CO_2} N_{CO_2} . \quad (2)$$

By substituting the values of all the physical parameters into (2), verify that for the atmosphere on Venus $\varepsilon \approx 1$. In this regime, find a simple approximation to (2) that relates linearly ε and $\alpha_{CO_2} N_{CO_2}$. Then, find a simple linear approximation to $n(h)$ with respect to the elevation h.

iii) (10 points) Finally, use the following well known result from the geometric optics for the radius of curvature of the light beam propagating parallel to the planet surface that has an atmosphere with a gradient of the refractive index. Then, verify that $|r_{light}| < r_V$:

$$r_{light} = \frac{n}{\partial n / \partial h} . \quad (3)$$

Question 3 (25 points). Line of sight IR communication system.



Consider a short-range laser communication system operating at the near Infra-Red (IR) carrier wavelength of $\lambda_c = 850nm$. The transmitter system features a lens with the focal distance $f_0 = 15cm$ that collimates emission from the VCSEL optical transmitter into a slowly divergent beam. The size of the VCSEL beam at the diode surface is $d_0 = 0.5mm$. The emitted power associated with the carrier wave is $P_0 = 0.6W$. After propagation distance L, the remaining beam is focused by the lens of diameter $D_l = 3cm$ onto a low-noise Si Avalanche Photodiode with photosensitivity of $r = 0.24\,A/W$, and a cutoff frequency of 700 *MHz*. Noise level of the photodetector is $I_n = 100\,pA$. There are two principal loss mechanisms in this system. One is due to variable atmospheric absorption of the carrier wave. Particularly, absorption of the IR radiation in humid air can be estimated as $\alpha_{air} = (1 + 30 \cdot h)\ [dB/km]$, where h is humidity concentration (*h=0* – dry air, *h=1* – 100% humidity). Another loss mechanism is due to divergence of the IR beam because of the imperfect collimation by the transmitter. Particularly, after propagation over a distance L from the transmitter, beam diameter increases and can be approximated as $D(L) \approx L \cdot d_0 / f_0$. In what follows answer the following questions.

i) (20 points) Assuming that the VCSEL is operating well below its cutoff frequency, estimate the maximal propagation distances L_{dry}, L_{humid} of the light beam on a dry ($h=0$) day, and on a very humid day ($h=1$) so that the signal-to-noise ratio on the receiver side remains higher than $23dB$. This condition guarantees the bit-error rate superior to 10^{-9}, which is a common standard in communication systems.

Hint: to find an estimate for the maximal propagation distance you will have to find an approximate solution of a certain nonlinear equation in the following general form $A - B \cdot \log(L) - C \cdot L = 0$. The simplest way of solving such an equation is via consecutive approximations. Thus, start with low values (say $L=1000m$) and then increase L in the increments of ~100 meters until an approximate solution is found.

ii) (5 points) If the desired link length is $L=5000m$, find the range of humidity values as which signal-to-noise ratio on the receiver side remains higher than $23dB$.

Question 4 (45 points). TM-polarized modes of a planar metallic waveguide.

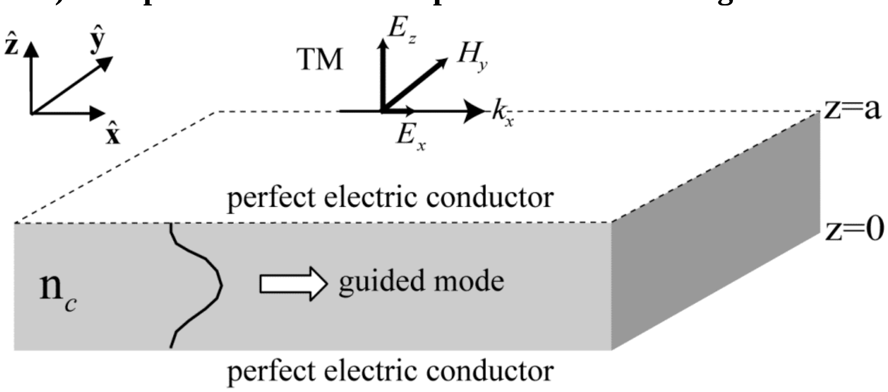


Fig. 1 Schematic of a planar waveguide with metallic boundaries

Consider a planar waveguide comprised of two infinite metallic plates positioned parallel to each other with separation a between them. The gap between two metal plates is filled with lossless dielectric of refractive index n_c (see Fig. 1). The dielectric-filled gap represents the core of a planar waveguide. Modes propagating in such a waveguide can be classified as either TE or TM. Particularly, the TM-polarized mode propagating along the X axis has only one component of the magnetic field that is directed along the Y axis: $\bar{H}_{TM} = \left(0, H_y(x,z,t), 0\right)$. Moreover, at the interfaces with the perfectly conductive metal plates, the electric field component that is parallel to the metal plates satisfies the following boundary conditions:

$$E_x(x,0,t) = E_x(x,a,t) = 0 \,. \quad (1)$$

In the case of TM polarization, modal magnetic field also satisfies Maxwell's equations that can be presented in the following simple form:

$$\frac{\partial^2 H_y(x,z,t)}{\partial x^2} + \frac{\partial^2 H_y(x,z,t)}{\partial z^2} - \frac{1}{C^2}\frac{\partial^2 H_y(x,z,t)}{\partial t^2} = 0 \,, \quad (2)$$

where the speed of light in the dielectric core is given by $C = c/n_c$, where c is the speed of light in vacuum.

Complex solution of Maxwell's equations (2) that describes waveguide mode of frequency ω propagating in the $\hat{\mathbf{x}}$ direction can be written as:

$$H_y(x,z,t) = h_y(z) \cdot \exp(i(k_x x - \omega t)), \quad (3)$$

where one can suppose that the function $h_y(z)$ is purely real. If the modal magnetic field $H_y(x,z,t)$ is known, then the modal electric field:

$$\bar{E}(x,z,t) = \left(e_x(z), 0, e_z(z)\right) \cdot \exp(i(k_x x - \omega t)), \quad (4)$$

can be found using one of the four Maxwell's equations (the Maxwell-Ampère's law $\nabla \times \bar{H} = \varepsilon_0 \varepsilon \, \partial \bar{E} / \partial t$, for example). Naturally, one has to ensure that the modal electric field (4) satisfies the boundary conditions (1).

<u>TM Modes of a planar metallic waveguide.</u>

Using the general form for the modal electric fields (3), (4), as well as Maxwell's equations, and boundary conditions (1) find the following:

(i) (10 points) Functional form of the modal magnetic field component $h_y(z)$ (it should respect both the Maxwell's equations and the boundary conditions).

(ii) (10 points) Functional form of the modal electric field components $e_x(z), e_z(z)$ that correspond to the modal magnetic field found in (i) (they should respect both the Maxwell's equations and the boundary conditions).

(iii) (10 points) Functional form of the modal dispersion relation (relation between k_x and ω).

<u>Single mode versus multi-mode propagation regimes.</u>

Typically, there are two types of solutions described by (3). Particularly, if the value of k_x is purely real, then $\mathrm{Re}\left(h_y(z)\exp(ik_x x - i\omega t)\right) = h_y(z)\cos(k_x x - \omega t)$ is a physical solution that defines a guided mode. However, if k_x is purely imaginary, then $\mathrm{Re}\left(h_y(z)\exp(ik_x x - i\omega t)\right) = h_y(z)\cos(\omega t)\exp(-|k_x| x)$ which defines an evanescent mode. At a given operation frequency ω, energy transport by the waveguide is possible only if there exists at least one guided mode (meaning the mode characterized with a purely real value of k_x). Finally, if at a given operation frequency ω there exist $n \geq 2$ modes, all characterized by the real values of k_x, then one sais that such a waveguide is multimode at that frequency and it supports *n* guided modes.

(iv) (5 points) Find the cut-off frequency ω_1 of the fundamental guided mode so that at lower frequencies $\omega < \omega_1$ the planar metallic waveguide cannot transport any energy.

(v) (5 points) For a given frequency ω, what is the number of guided modes supported by the planar metallic waveguide of the core size *a* and the core refractive index n_c?

(vi) (5 points) In the case of a waveguide of the core size $a = 500\mu m$ filled with dry gas of refractive index $n_c \approx 1$ find the number of guided modes supported by such a waveguide if the operation frequency is $\omega = 1THz$.