# ORDRE DES INGÉNIEURS DU QUÉBEC

# NOVEMBRE 2022 SESSION

Open book examination
Calculator : authorized model only
Duration : 3 hours

# 20-MB-B13 ADVANCED MATHEMATICS

**Question 1. (20 points)**

For each of the following differential equations, develop the solution in a power series of $x$, and derive a simple expression of the solution from it.

a) [10 pts]

$$y'(x) - 2xy(x) = 0, \quad y(0) = 1.$$

b) [10 pts]

$$y''(x) - xy'(x) - y(x) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

**Question 2. (20 points)**

Find the Fourier series coefficients of the following $2\pi$-periodic functions.

a) [10 pts]

$$\begin{aligned} f(x) = & \quad -1, \quad \text{when} \quad -\pi < x \leq 0, \\ f(x) = & \quad 1, \quad \text{when} \quad 0 < x \leq \pi. \end{aligned}$$

b) [10 pts]

$$f(x) = x + \pi, \quad \text{when} \quad -\pi < x \leq \pi$$

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**Question 3. (20 points)**

Let $f(x) = e^x - 3x$ and let us consider the equation $f(x) = 0$.

(a) [5 pts] Prove that there is a root in the interval [0, 1]. Why can we say that there is only one in this interval ?

(b) [5 pts] Can we apply the bisection method starting from the interval [0, 1] ? If yes, how many iterations are needed for the error to be less than $10^{-8}$ ?

(c) [5 pts] Without computing any iteration, determine if the fixed point method $x_{n+1} = g(x_n)$ with $g(x) = \frac{1}{3}e^x$ converge. What is the convergence order ?

(d) [5 pts] Without computing any iteration, determine if Newton's method applied to this equation is of quadratic (at least) or first order.

**Question 4. (20 points)**

Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) [10 pts] Does A admit a Cholesky factorization ?

(b) [10 pts] Compute (showing the details of the computations) the matrix $\tilde{U}$ of the Crout factorization ($A = L\tilde{U}$) knowing that

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Question 5. (20 points)**

Show that the following function interpolates $(-1, 4)$, $(0, 0)$ et $(1, 0)$, that it is a cubic spline and that it satisfies free (natural) conditions, for coefficients $a$, $b$ et $c$ that you will determine.

$$S(x) = \begin{cases} S_0(x) = a + b(x+1) + c(x+1)^3 \text{ pour } -1 \leq x \leq 0, \\ S_1(x) = -2x + 3x^2 - x^3 \text{ pour } 0 \leq x \leq 1. \end{cases}$$